Optimal Control of the Thermistor Problem

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The presented talk deals with the optimal control of the thermistor problem that models the conductive heat transfer in a conductor produced by an electric current. This leads to the following quasi-linear system of partial differential equations (PDEs):

$$\partial_t \theta - \operatorname{div}(\kappa \nabla \theta) = (\sigma(\theta)\nabla\varphi) \cdot \nabla\varphi \quad \text{in Q} \tag{1}$$

$$\nu \cdot \kappa \nabla \theta + \alpha\theta = \alpha\theta_l \quad \text{on}\Sigma \tag{2}$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega \tag{3}$$

$$-\operatorname{div}(\sigma(\theta)\nabla\varphi) = 0 \quad \text{in } Q \tag{4}$$

$$\nu \cdot \sigma(\theta)\nabla\varphi = u \quad \text{on } \Sigma_0 \tag{5}$$

$$\varphi = 0 \quad \text{on } \Sigma \backslash \Sigma_0, \tag{6}$$

with a Lipschitz domain $\Omega \subset \mathbf{R}^2$, $Q = \Omega \times]0,T[$, $\Sigma = \partial\Omega \times]0,T[$, and $\Sigma_0 = \Gamma_0 \times]0,T[$, where Γ_0 denotes a fixed part of $\partial\Omega$. Moreover, θ represents the temperature, while φ is the electric potential. Furthermore, θ_l and θ_0 are given functions, and u is the control that can be interpreted as a current induced on Γ_0. A possible application for this coupled system of PDEs is the hardening of steel workpieces via the Joule effect.

Our aim is to adjust the control u such that

$$J(\theta, u) := \frac{1}{2} \|\theta(T) - \theta_d\|^2_{L^2(\Omega_m)} + \frac{\beta}{2} \|u\|^2_{L^2(\Sigma_0)} \tag{7}$$

is minimized subject to (1)–(6) and the following inequality constraints

$$u_a \le u(t,x) \le u_b \quad \text{a.e. on } \Sigma_0 \tag{8}$$

$$\theta_a(t,x) \le \theta(t,x) \le \theta_b(t,x) \quad \text{a.e. in } Q. \tag{9}$$

Here, (8) reflects the maximum available electrical power, whereas (9) prevents melting of the material which is crucial in view of hardening applications. Notice that (9) represents a pointwise state constraint that is known to be numerically and theoretically challenging to handle. To be more precise, the generalized Karush-Kuhn-Tucker theory requires to consider the state constraints in the space of continuous functions. The continuity of solutions to (1)–(6) is shown by using maximum parabolic regularity results. After stating optimality conditions, we will turn to the numerical treatment of this optimization problem by means of a Moreau-Yosida type regularizaton of the state constraints. The feasibility of this approach is afterwards demonstrated by the example of hardening a gear rack used in the automotive industry.